THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Suggested Solution to Final Examination

1. (a) (i) Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

(ii) Let $u, v \in [1, \infty)$ and $u < v$. Since $f(x) = \frac{1}{x^2}$ is continuous on $[u, v]$ and differentiable on (u, v), by the Mean Value Theorem, there exists $c \in (u, v)$ such that

$$\frac{f(v) - f(u)}{v - u} = f'(c) = -\frac{2}{c^3}.$$

Then,

$$\left|\frac{f(v) - f(u)}{v - u}\right| = \left|-\frac{2}{c^3}\right| = \frac{2}{c^3} \leq 2$$

and so

$$|f(v) - f(u)| \leq 2|v - u|.$$

f is a Lipschitz function on $[1, \infty)$ and therefore uniformly continuous on $[1, \infty)$.

(b) Let $v_n = \frac{1}{2n} \geq 0$ and $u_n = \frac{1}{n} \geq 0$. Then $\lim_{n\to\infty} v_n - u_n = -\frac{1}{2n} = 0$, however $\lim_{n\to\infty} f(v_n) - f(u_n) = \lim_{n\to\infty} 3n^2$ which goes to positive infinity. Therefore, $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, \infty)$.

2. Let $\epsilon > 0$. Since $f : [a, b] \to \mathbb{R}$ is continuous and hence uniformly continuous, there exists $\delta > 0$ such that for any $u, v \in [a, b]$ and $|u - v| < \delta$, then $|f(u) - f(v)| < \frac{\epsilon}{2}$.

By Archimedean property, there exists $m \in \mathbb{N}$ such that $h := \frac{b - a}{m} < \delta$. Define $I_1 = [a, a + h]$ and $I_k = (a + (k - 1)h, a + kh]$ for $k = 2, 3, \cdots, m$. On each interval I_k, define g to be the linear function joining the points

$$(a + (k - 1)h, f(a + (k - 1)h)) \qquad \text{and} \qquad (a + kh, f(a + kh)).$$

Then g is a continuous piecewise linear function on $[a, b]$.

Let $x \in I_k$. By the construction of g, $f(a + kh) = g(a + kh)$, then

$$\begin{aligned}
|f(x) - g(x)| &= |f(x) - f(a + kh) + g(a + kh) - g(x)| \\
&\geq |f(x) - f(a + kh)| + |g(a + kh) - g(x)| \\
&\geq |f(x) - f(a + kh)| + |g(a + kh) - g(a + (k - 1)h)| \\
&= |f(x) - f(a + kh)| + |f(a + kh) - f(a + (k - 1)h)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

3. Let $\epsilon > 0$. Let $P = \{0, 1 - \frac{\epsilon}{3}, 1, 2 - \frac{\epsilon}{3}, 2\}$ be a partition of the interval $[0, 2]$. Then we have

$$\begin{aligned} U(P,f) &= (1)(1-\frac{\epsilon}{3}) + (2)(\frac{\epsilon}{3}) + (2)(1-\frac{\epsilon}{3}) + (3)(\frac{\epsilon}{3}) \\ L(P,f) &= (1)(1-\frac{\epsilon}{3}) + (1)(\frac{\epsilon}{3}) + (2)(1-\frac{\epsilon}{3}) + (2)(\frac{\epsilon}{3}) \end{aligned}$$

Therefore, $U(P,f) - L(P,f) = \dfrac{2\epsilon}{3} < \epsilon$ and f is integrable on $[0, 2]$.

4. Note that $\inf f([x_{i-1}, x_i]) \leq f(a + (i - \frac{1}{2})h_n) \leq \sup f([x_{i-1}, x_i])$, so

$$L(P,f) \leq M_n(P,f) \leq U(P,f).$$

Since f is integrable on $[a, b]$, $\lim_{n\to\infty} L(P,f) = \lim_{n\to\infty} U(P,f) = \int_a^b f$, by using the sandwich theorem, $\lim_{n\to\infty} M_n(P,f) = \int_a^b f$.

5. (a) Define $f : [a, b] \to \mathbb{R}$ by $f(a) = 1$ but $f(x) = 0$ for $a < x \leq b$. Then $\int_a^b f = 0$, but f is not the zero function.

   (b) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function but not the zero function, then there exists $x_0 \in [a, b]$ such that $f(x_0) > 0$.
   Since f is continuous at x_0, let $\epsilon_0 = \dfrac{f(x_0)}{2} > 0$, there exists $\delta > 0$ such that for all $x \in [a, b]$ with $|x - x_0| < \delta$,

$$-\frac{f(x_0)}{2} = -\epsilon < f(x) - f(x_0) < \epsilon = \frac{f(x_0)}{2}.$$

   Then $f(x) > \dfrac{f(x_0)}{2} > 0$.
   Now, take $x_1, x_2 \in [a, b] \cap (x_0 - \delta, x_0 + \delta)$ with $x_1 < x_2$ and choose a partition P such that x_1 and x_2 are partition points of P. Then, we have

$$L(P,f) \geq \inf f([x_1, x_2]) \cdot (x_2 - x_1) \geq \frac{f(x_0)}{2} \cdot (x_2 - x_1) > 0.$$

   Then $\int_a^b f \geq U(P,f) > 0$.

6.
- If $x, y \in \mathbb{R}^+$, then clearly $d(x,y) = \left|\ln\left(\frac{y}{x}\right)\right| \geq 0$. Furthermore, if $d(x,y) = \left|\ln\left(\frac{y}{x}\right)\right| = 0$, then $\dfrac{y}{x} = 1$ and so $x = y$.
- If $x, y \in \mathbb{R}^+$, then

$$d(y,x) = \left|\ln\left(\frac{x}{y}\right)\right| = |\ln x - \ln y| = |\ln y - \ln x| = \left|\ln\left(\frac{y}{x}\right)\right| = d(x,y).$$

- If $x, y, z \in \mathbb{R}^+$, then

$$\begin{aligned} d(x,y) + d(y,z) &= \left|\ln\left(\frac{y}{x}\right)\right| + \left|\ln\left(\frac{z}{y}\right)\right| \\ &= |\ln y - \ln x| + |\ln z - \ln y| \\ &\geq |\ln z - \ln x| \qquad \text{(triangle inequality)} \\ &= \left|\ln\left(\frac{z}{x}\right)\right| \\ &= d(z,x) \end{aligned}$$

7.
- Let $\{x_n\} \in X$ and $a \in \mathbb{R}$. We have

$$\|(a \cdot \{x_n\})\| = \|\{ax_n\}\| = \sup\{|ax_1|, |ax_2|, \cdots\} = |a| \sup\{|x_1|, |x_2|, \cdots\} = |a| \cdot \|\{x_n\}\|.$$

- Let $\{x_n\}, \{y_n\} \in X$. Since $\|\{x_n\}\| \geq |x_i|$ and $\|\{y_n\}\| \geq |y_i|$ for all $i \in \mathbb{N}$, $\|\{x_n\}\| + \|\{y_n\}\| \geq |x_i| + |y_i| \geq |x_i + y_i|$ for all $i \in \mathbb{N}$. Hence, we have

$$\|\{x_n\}\| + \|\{y_n\}\| \geq \sup\{|x_1 + y_1|, |x_2 + y_2|, \cdots\} = \|\{x_n\} + \{y_n\}\|.$$

- Let $\{x_n\} \in X$ and $\|\{x_n\}\| = 0$. $0 \geq \|\{x_n\}\| \geq |x_i| \geq 0$ for all $i \in \mathbb{N}$ which implies that $x_i = 0$ for all $i \in \mathbb{N}$.

8. (a) Let r_0 be a rational number in $[0, 1]$, then $f(r_0)$ is nonzero. Since the set of irrational numbers in $[0, 1]$ is dense in $[0, 1]$, there exists an irrational sequence $\{x_n\}$ in $[0, 1]$ such that $\lim_{n\to\infty} x_n = r_0$. However, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \neq f(r_0)$. Therefore, f is discontinuous at every rational number in $[0, 1]$.

(b) Let x_0 be an irrational number in $[0, 1]$ and let $\epsilon > 0$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Since there are only finitely many rational numbers with denominator less than N in the interval $[0, 1]$, we can choose $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta)$ is a subset of $[0, 1]$ and it contains no rational numbers with denominator less than N. It follows that for $|x - x_0| < \delta$,

- if x is rational, $x = \frac{m}{n}$ where m and n are natural numbers with $\gcd(m, n) = 1$ and $n \geq N$, then $|f(x) - f(x_0)| = \frac{1}{n} - 0 \leq \frac{1}{N} < \epsilon$;
- if x is irrational, then $|f(x) - f(x_0)| = 0 - 0 = 0 < \epsilon$.

Therefore, f is continuous at every irrational number in $[0, 1]$.

(c) Let $\epsilon > 0$. By the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. There are only finitely many rational numbers with denominator less than N in the interval $(0, 1)$, we label them as $x_1, x_2, \cdots, x_{n-1}$ such that $x_0 := 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n := 1$. We choose a sufficiently small $\delta > 0$ such that $\delta < \frac{\epsilon}{4n}$ (i.e. $2n\delta < \frac{\epsilon}{2}$) and

$$x_0 < x_0 + \delta < x_1 - \delta < x_1 + \delta < x_2 - \delta < x_2 + \delta < \cdots < x_{n-1} - \delta < x_{n-1} + \delta < x_n - \delta < x_n.$$

We let $P = \{x_0, x_0 + \delta, x_1 - \delta, x_1 + \delta, x_2 - \delta, x_2 + \delta \cdots, x_{n-1} - \delta, x_{n-1} + \delta, x_n - \delta, x_n\}$ be a partition of $[0, 1]$. Note that $L(P, f) = 0$ as every subinterval contains an irrational number.

Also, we have

$$\begin{aligned}
& U(P,f) \\
= \;& \delta\cdot\sup f([x_0,x_0+\delta])+(x_1-x_0-2\delta)\cdot\sup f([x_0+\delta,x_1-\delta]) \\
& +2\delta\cdot\sup f([x_1-\delta,x_1+\delta])+\cdots \\
& +(x_n-x_{n-1}-2\delta)\cdot\sup f([x_{n-1}+\delta,x_n-\delta])+\delta\cdot\sup f([x_n-\delta,x_n]) \\
= \;& \delta\cdot(\sup f([x_0,x_0+\delta])+\sup f([x_n-\delta,x_n])) \\
& +\sum_{i=1}^{n}(x_i-x_{i-1}-2\delta)\cdot\sup f([x_i+\delta,x_{i-1}-\delta])+2\delta\sum_{i=1}^{n-1}\sup f([x_1-\delta,x_1+\delta]) \\
\le \;& \delta\cdot(1+1)+\sum_{i=1}^{n}(x_i-x_{i-1}-2\delta)\cdot\frac{1}{N}+2\delta\sum_{i=1}^{n-1}1 \\
= \;& 2n\delta+\frac{1-2n\delta}{N} \\
\le \;& 2n\delta+\frac{1}{N} \\
< \;& \frac{\epsilon}{2}+\frac{\epsilon}{2} \\
= \;& \epsilon
\end{aligned}$$

Therefore, $U(P,f)-L(P,f)<\epsilon$ and f is integrable on $[0,1]$.